

# The failure of the uncountable non-commutative Specker Phenomenon

By

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## Abstract

Higman proved in 1952 that every free group is non-commutatively slender, this is to say that if  $G$  is a free group and  $h$  is a homomorphism from the countable complete free product  $\mathfrak{x}_\omega \mathbb{Z}$  to  $G$ , then there exists a finite subset  $F \subseteq \omega$  and a homomorphism  $\bar{h} : \ast_{i \in F} \mathbb{Z} \rightarrow G$  such that  $h = \bar{h} \rho_F$ , where  $\rho_F$  is the natural map from  $\mathfrak{x}_{i \in \omega} \mathbb{Z}$  to  $\ast_{i \in F} \mathbb{Z}$ . Corresponding to the abelian case this phenomenon was called the non-commutative Specker Phenomenon. In this paper we show that Higman's result fails if one passes from countable to uncountable. In particular, we show that for non-trivial groups  $G_\alpha$  ( $\alpha \in \lambda$ ) and uncountable cardinal  $\lambda$  there are  $2^{2^\lambda}$  homomorphisms from the complete free product of the  $G_\alpha$ 's to the ring of integers.

## 0 Introduction

Higman proved in 1952 [H] that every free group  $F$  is non-commutatively slender, where slenderness means that any homomorphism  $h$  from the countable complete free product  $\mathfrak{x}_{i \in \omega} \mathbb{Z}$  of the integers to  $F$  depends only on finitely many coordinates. A similar result was proved by Specker in 1950 [S] for abelian groups. Specker showed that any homomorphism from the countable product  $\Pi_\omega \mathbb{Z}$  to the integers depends only on finitely many entries. These two phenomena were called the commutative and the non-commutative Specker Phenomenon. Eda extended Higman's result in 1992 in [E1] by showing that for any non-commutatively slender group  $S$ , non-trivial groups  $G_\alpha$  ( $\alpha \in I$ ) and any

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homomorphism  $h$  from the free  $\sigma$ -product of the  $G'_\alpha$ 's to  $\mathbb{Z}$  there exist a finite subset  $F$  of  $I$  and a homomorphism  $\bar{h} : *_{i \in F} G_i \rightarrow S$  such that  $h = \bar{h} \rho_F$ , where  $\rho_F$  is the natural map from  $\mathfrak{X}_{i \in I}^\sigma G_i$  to  $*_{i \in F} G_i$ . Motivated by this result Eda asked the question [E1][Question 3.8] whether the non-commutative Specker Phenomenon still holds if one passes from countable to uncountable. In this paper we will answer this question to the negative by constructing for a given uncountable cardinal  $\lambda$  and non-trivial groups  $G_\alpha$  ( $\alpha \in \lambda$ ), a homomorphism  $h$  from the complete free product of the  $G_\alpha$ 's to  $\mathbb{Z}$  for which the non-commutative Specker Phenomenon fails. In particular, we will show that there are  $2^{2^\lambda}$  of these homomorphisms, hence the size of the set of all homomorphisms from  $\mathfrak{X}_{\alpha \in \lambda} G_\alpha$  to the integers is as large as possible.

## 1 Basics and notations

Let  $I$  be an arbitrary indexset. For groups  $G_i$  ( $i \in I$ ), the free product is denoted by  $*_{i \in I} G_i$  (see [M] for details on free products). If  $J$  is a finite subset of  $I$  then we write  $J \Subset I$ . For  $X \subset Y \subset I$  let  $\rho_{XY} : *_{i \in Y} G_i \rightarrow *_{i \in X} G_i$  be the canonical homomorphism. Then, the set  $\{*_{i \in X} G_i : X \Subset I\}$  together with the homomorphisms  $\rho_{XY}$  ( $X \subset Y \Subset I$ ) form an inverse system and its inverse limit  $\varprojlim (*_{i \in X} G_i, \rho_{XY} : X \subset Y \Subset I)$  is called the *unrestricted free product* (see [H]). Eda [E1] introduced an infinite version of free products and defined the *complete free product*  $\mathfrak{X}_{i \in I} G_i$  of the groups  $G_i$  which is isomorphic to the subgroup  $\bigcap_{F \Subset I} \{*_{i \in F} G_i * \varprojlim (*_{i \in X} G_i, \rho_{XY} : X \subset Y \Subset I)\}$  of the unrestricted free product. To get familiar with the complete free product we recall the definition of words of infinite length and some basic facts about  $\mathfrak{X}_{i \in I} G_i$  from [E1].

**Definition 1.1** Let  $G_i$  ( $i \in I$ ) be non-trivial groups such that  $G_i \cap G_j = \{e\}$  for  $i \neq j \in I$ . Elements of  $\bigcup_{i \in I} G_i$  are called *letters*. A *word*  $W$  is a function  $W : \bar{W} \rightarrow \bigcup_{i \in I} G_i$  such that  $\bar{W}$  is a linearly ordered set and  $W^{-1}(G_i)$  is finite for any  $i \in I$ . In case the cardinality of  $\bar{W}$  is countable, we say that  $W$  is a  *$\sigma$ -word*. The class of all words is denoted by  $\mathcal{W}(G_i : i \in I)$  (abbreviated by  $\mathcal{W}$ ) and the class of all  $\sigma$ -words is denoted by  $\mathcal{W}^\sigma(G_i : i \in I)$  (abbreviated by  $\mathcal{W}^\sigma$ ).

Two words  $U$  and  $V$  are said to be *isomorphic* ( $U \cong V$ ) if there exists an isomorphism  $\varphi : \bar{U} \rightarrow \bar{V}$  as linearly ordered sets such that  $U(\alpha) = V(\varphi(\alpha))$  for all  $\alpha \in \bar{U}$ . It is easily seen that  $\mathcal{W}$  is a set and that for words of finite length the above definition coincides with the usual definition of words. For a subset  $X \subset I$  the *restricted word* (or *subword*)  $W_X$  of  $W$  is given by the

function  $W_X : \bar{W}_X \rightarrow \bigcup_{i \in X} G_i$ , where  $\bar{W}_X = \{\alpha \in \bar{W} : W(\alpha) \in \bigcup_{i \in X} G_i\}$  and  $W_X(\alpha) = W(\alpha)$  for all  $\alpha \in \bar{W}_X$ . Hence  $W_X \in \mathcal{W}$ . Now an equivalence relation is defined on  $\mathcal{W}$  by saying that two words  $U$  and  $V$  are *equivalent* ( $U \sim V$ ) if  $U_F = V_F$  for all  $F \subseteq I$ , where we regard  $U_F$  and  $V_F$  as elements of the free product  $*_{i \in F} G_i$ . The equivalence class of a word  $W$  is denoted by  $[W]$  and the composition of two words as well as the inverse of a word are defined natural. Thus  $\mathcal{W}/\sim = \{[W] : W \in \mathcal{W}\}$  becomes a group.

**Definition 1.2** The *complete free product*  $\mathfrak{x}_{i \in I} G_i$  is the group  $\mathcal{W}(G_i : i \in I)/\sim$ . The *free  $\sigma$ -product*  $\mathfrak{x}_{i \in I}^\sigma G_i$  is the group  $\mathcal{W}^\sigma(G_i : i \in I)/\sim$ , which is a subgroup of  $\mathfrak{x}_{i \in I} G_i$ . In case every  $G_i$  is isomorphic to  $G$ , we abbreviate  $\mathfrak{x}_{i \in I} G_i$  by  $\mathfrak{x}_I G$  and similarly for free  $\sigma$ -products.

Obviously,  $\mathfrak{x}_{i \in I} G_i$  and  $\mathfrak{x}_{i \in I}^\sigma G_i$  are isomorphic to  $*_{i \in I} G_i$  if  $I$  is finite. By [E1, Proposition 1.8] the complete free product  $\mathfrak{x}_{i \in I} G_i$  is isomorphic to the subgroup  $\bigcap_{F \subseteq I} \{ *_{i \in F} G_i * \lim_{\leftarrow} (*_{i \in X} G_i, \rho_{XY} : X \subset Y \subseteq I) \}$  of the unrestricted free product. Moreover, Eda proved in [E1] that each equivalence class  $[W]$  is determined uniquely by a reduced word. A word  $W \in \mathcal{W}(G_i : i \in I)$  is called *reduced*, if  $W \cong UXV$  implies  $[X] \neq e$  for any non-empty word  $X$ , where  $e$  is the identity, and for any neighboring elements  $\alpha$  and  $\beta$  of  $\bar{W}$  it never occurs that  $W(\alpha)$  and  $W(\beta)$  belong to the same  $G_i$ .

**Lemma 1.3 (Eda, [E1])** *For any word  $W \in \mathcal{W}(G_i : i \in I)$ , there exists a reduced word  $V \in \mathcal{W}(G_i : i \in I)$  such that  $[W] = [V]$  and  $V$  is unique up to isomorphism.*

Furthermore, Eda showed in [E1] the following lemma where a word  $W \in \mathcal{W}(G_i : i \in I)$  is called *quasi-reduced* if the reduced word of  $W$  can be obtained by multiplying neighboring elements without cancelation.

**Lemma 1.4 (Eda, [E1])** *For any two reduced words  $W, V \in \mathcal{W}(G_i : i \in I)$  there exist reduced words  $V_1, W_1, M \in \mathcal{W}(G_i : i \in I)$  such that  $W \cong W_1 M$ ,  $V \cong M^{-1} V_1$  and  $W_1 V_1$  is quasi-reduced.*

We would like to remark that a free  $\sigma$ -product  $\mathfrak{x}_{i \in I}^\sigma \mathbb{Z}_i$  is isomorphic to the fundamental group and the free complete product  $\mathfrak{x}_{i \in I} \mathbb{Z}_i$  is isomorphic to the big fundamental group of the Hawaiian earring with  $I$ -many circles (see [CC]). Hence free complete products are also of topological interest.

## 2 The uncountable Specker Phenomenon

In 1950, E. Specker [S] proved that for any homomorphism  $h$  from the countable direct product  $\mathbb{Z}^\omega$  to the ring of integers  $\mathbb{Z}$ , there exist a finite subset  $F$  of  $\omega$

and a homomorphism  $\bar{h} : \mathbb{Z}^F \rightarrow \mathbb{Z}$  satisfying  $h = \bar{h}\rho_F$ , where  $\rho_F : \mathbb{Z}^\omega \rightarrow \mathbb{Z}^F$  is the canonical projection. This phenomenon is called the Specker Phenomenon and it can easily be seen that Specker's result still holds if one replaces  $h : \mathbb{Z}^\omega \rightarrow \mathbb{Z}$  by  $g : \mathbb{Z}^\omega \rightarrow G$ , where  $G$  is any free abelian group. For generalizations to products of larger cardinalities and the resulting definition of slenderness for abelian groups we refer to [EM] or [F1]. In [E2] Eda introduced a non-commutative version of slenderness.

**Definition 2.1** A group  $G$  is *non-commutatively slender* if for any homomorphism  $h : \mathfrak{X}_{\mathbb{N}}\mathbb{Z} \rightarrow G$  there exists a natural number  $n$  such that  $h(\mathfrak{X}_{\mathbb{N} \setminus \{1, \dots, n\}}\mathbb{Z}) = \{e\}$ .

Eda proved that every non-commutatively slender group is torsion-free and that in the abelian case non-commutative slenderness is equivalent to the commutative slenderness (see [E1, Theorem 3.3. and Corollary 3.4.]). Moreover, he proved that non-commutative slender groups behave nice in the following sense for non-trivial groups  $G_i$  ( $i \in I$ ).

**Proposition 2.2 (Eda, [E1])** *Let  $S$  be a non-commutative slender group and  $h : \mathfrak{X}_{i \in I}^\sigma G_i \rightarrow S$  be a homomorphism. Then, there exist a finite subset  $F$  of  $I$  and a homomorphism  $\bar{h} : \mathfrak{X}_{i \in F} G_i \rightarrow S$  such that  $h = \bar{h}\rho_F$ , where  $\rho_F$  is the natural map from  $\mathfrak{X}_{i \in I}^\sigma G_i$  to  $\mathfrak{X}_{i \in F} G_i$ .*

Moreover, if  $S_j$  ( $j \in J$ ) are non-commutatively slender groups then also the restricted direct product and the free product of the  $S_j$ 's are non-commutatively slender (see [E1, Theorem 3.6.]). The first fundamental result on the class of non-commutatively slender groups was already obtained by Higman in [H] where he proved the following theorem.

**Theorem 2.3 (Higman, [H])** *Every free group is non-commutatively slender.*

In contrast to Higman's result we will show that if one replaces countable by uncountable then the non-commutative Specker Phenomenon fails. In particular we show that there are  $2^{2^\lambda}$  homomorphisms from the complete free product  $\mathfrak{X}_{\alpha \in \lambda} G_\alpha$  to the ring of integers if  $\lambda$  is uncountable and the  $G_\alpha$ 's are non-trivial groups. For the convenience of the reader we first construct one homomorphism for which the Specker Phenomenon fails and then modify the construction to obtain our main result.

**Theorem 2.4** *Let  $\lambda$  be any uncountable cardinal and  $G_\alpha$  ( $\alpha \in \lambda$ ) non-trivial groups. For each  $\kappa \leq \lambda$  regular, uncountable there exists a homomorphism  $\varphi_\kappa : \mathfrak{X}_{\alpha \in \lambda} G_\alpha \rightarrow \mathbb{Z}$  for which the Specker Phenomenon fails.*

**Proof.** Let  $G_\alpha$  ( $\alpha < \lambda$ ) be a collection of non-trivial groups and choose  $e_\alpha \neq g_\alpha \in G_\alpha$ , where  $e_\alpha$  is the identity of  $G_\alpha$  ( $\alpha < \lambda$ ). We define the following words  $M_\kappa \in \mathfrak{X}_{\alpha \in \lambda} G_\alpha$  for any regular, uncountable cardinal  $\kappa < \lambda$ .

$$M_\kappa : (\kappa, <) \longrightarrow \cup_{\alpha < \lambda} G_\alpha \text{ via } \beta \mapsto g_\beta$$

where  $<$  is the natural ordering of  $\lambda$ . Note that  $M_\kappa$  is a word of uncountable cofinality since  $\kappa$  is regular and uncountable. For  $\beta < \kappa$  we let  $M_{\kappa, \beta}$  be the subword  $M_\kappa \upharpoonright_{[\beta, \kappa)}$  of  $M_\kappa$ . Now let  $X$  be any reduced word in  $\mathfrak{X}_{\alpha \in \lambda} G_\alpha$  and recall that a subset  $J \subseteq (\lambda, <)$  is called convex if  $x < y < z$  and  $x, z \in J$  implies  $y \in J$ . We put

$$Occ_\kappa^+(X) := \{J \subseteq (\lambda, <) : J \text{ is convex and } X \upharpoonright_J \cong M_{\kappa, \beta} \text{ for some } \beta < \kappa\}.$$

Thus  $Occ_\kappa^+(X)$  counts the occurrences of end segments of  $M_\kappa$  in  $X$ . Similarly we let

$$Occ_\kappa^-(X) := \{J \subseteq (\lambda, <) : J \text{ is convex and } X \upharpoonright_J \cong M_{\kappa, \beta}^{-1} \text{ for some } \beta < \kappa\}.$$

In order to avoid counting subsets of  $(\lambda, <)$  several times we define an equivalence relation on  $Occ_\kappa^+(X)$  and  $Occ_\kappa^-(X)$  in the following way. Two convex subsets  $J_1, J_2$  of  $(\lambda, <)$  are said to be equivalent if they have a common end segment, i.e.  $J_1 \sim_\kappa J_2$  if there exist  $j_i \in J_i$  such that  $X \upharpoonright_{S_1} \cong X \upharpoonright_{S_2}$ , where  $S_i = \{j \in J_i : j \geq j_i\}$ . First we prove that two subsets  $J_1, J_2 \in Occ_\kappa^+(X)$  are either disjoint or equivalent. Therefore assume that  $J_1, J_2 \in Occ_\kappa^+(X)$  are not disjoint, hence there exists  $j^* \in J_1 \cap J_2$ . We let  $h_i : M_{\kappa, \beta_i} \longrightarrow X \upharpoonright_{J_i}$  be isomorphisms for some  $\beta_i < \lambda$  ( $i = 1, 2$ ). Thus we can find  $\gamma_i \geq \beta_i$  such that  $h_i(\gamma_i) = j^*$  and therefore  $X(j^*) = g_{\gamma_i}$  for  $i = 1, 2$ . Hence  $\gamma_1 = \gamma_2$  and by transfinite induction we conclude  $X \upharpoonright_{T_1} \cong X \upharpoonright_{T_2}$ , where  $T_i = \{j \in J_i : j \geq j^*\}$ . Note that  $h_i$  is an isomorphism of linearly ordered sets, hence  $h_i$  commutes with limits and the successor-function. Similarly two subsets  $J_1, J_2 \in Occ_\kappa^-(X)$  are either disjoint or equivalent. Next we will show that the sets  $Occ_\kappa^+(X)/\sim_\kappa$  and  $Occ_\kappa^-(X)/\sim_\kappa$  are finite. Therefore assume that there exist infinitely many pairwise non-equivalent  $J_n \in Occ_\kappa^+(X)$  ( $n \in \omega$ ). Hence  $J_n$  and  $J_m$  are disjoint for  $n \neq m$ . We let  $X \upharpoonright_{J_n} \cong M_{\kappa, \beta_n}$  for some  $\beta_n < \kappa$  and  $n \in \omega$ . Then  $\beta = \cup_{n \in \omega} \beta_n$  is strictly less than  $\kappa$  since  $\kappa$  is regular and uncountable, hence  $cf(\kappa) > \aleph_0$ . Since  $\beta \in [\beta_n, \kappa)$  for all  $n \in \omega$  we can find  $j_n \in J_n$  such that

$$X(j_n) = M_{\kappa, \beta_n}(\beta) = M_{\kappa, \beta}(\beta).$$

for  $n \in \omega$ . But all  $J_n$  are pairwise disjoint and therefore  $X^{-1}(G_\beta)$  is infinite which is a contradiction. Thus  $Occ_\kappa^+(X)/\sim_\kappa$  and similarly  $Occ_\kappa^-(X)/\sim_\kappa$  are finite sets. We now define  $\varphi_\kappa : \mathfrak{X}_{\alpha \in \lambda} G_\alpha \longrightarrow \mathbb{Z}$  as follows:

$$X \mapsto |Occ_\kappa^+(X)/\sim_\kappa| - |Occ_\kappa^-(X)/\sim_\kappa|$$

where  $V$  is the reduced word corresponding to  $X$ .

Note that  $\varphi_\kappa$  is well-defined by Lemma 1.3. Moreover, by definition  $\varphi_\kappa(X^{-1}) = -\varphi_\kappa(X)$  and obviously the Specker Phenomenon fails for  $\varphi_\kappa$ . All we have to show is that  $\varphi_\kappa$  is a homomorphism. Therefore let  $X$  and  $Y$  be reduced words. By Lemma 1.4 there exist reduced words  $X_1, Y_1$  and  $M$  such that  $X \cong X_1 M$  and  $Y \cong M^{-1} Y_1$  and  $X_1 Y_1$  is quasi-reduced. Now it is easy to check that  $\varphi_\kappa(XY) = \varphi_\kappa(X_1 Y_1)$  by definition and hence

$$\varphi_\kappa(XY) = \varphi_\kappa(X_1 Y_1) = \varphi_\kappa(X_1) + \varphi_\kappa(Y_1) = \varphi_\kappa(X) + \varphi_\kappa(Y),$$

since  $X_1 Y_1$  is quasi-reduced.  $\square$

We would like to remark that the uncountable cofinality of  $\lambda$  in Theorem 2.4 is essential and can not be avoided by Higman's theorem. Modifying the proof of Theorem 2.4 we obtain

**Theorem 2.5** *Let  $\lambda$  be an uncountable cardinal and  $G_\alpha$  ( $\alpha \in \lambda$ ) be non-trivial groups. Then there are  $2^{2^\lambda}$  homomorphisms from the complete free product of the  $G_\alpha$ 's to the ring of integers.*

**Proof.** Let  $M_\alpha$  be a reduced word in  $\mathbb{X}_{\alpha \in \lambda} G_\alpha$  of uncountable cofinality  $\lambda$ , i.e.  $\bar{M}_\alpha = (\lambda, <)$ . Recall from the proof of Theorem 2.4 that by  $M_{\alpha, \beta}$  we mean the subword  $M_\alpha \upharpoonright_{[\beta, \lambda)}$  for  $\beta \in \lambda$ . Assume that we have a family of such words  $M_\alpha$  ( $\alpha \in 2^\lambda$ ) satisfying the following condition for a convex subset  $J \subseteq \lambda$  and a reduced word  $X$ :

$$X \upharpoonright_J \cong M_{\alpha, \beta} \text{ for } \beta \in \lambda \implies X \upharpoonright_J \not\cong M_{\gamma, \delta} \text{ for all } \alpha \neq \gamma \in 2^\lambda, \delta \in \lambda \quad (*)$$

Then it is well-known that we can choose  $2^{2^\lambda}$  almost disjoint families  $F_\alpha = \{M_\beta : \beta \in I_\alpha\}$  such that  $I_\alpha$  has size  $\lambda$ . We now define for  $\alpha \in 2^{2^\lambda}$

$$Occ_\alpha^+(X) := \{J \subseteq (\lambda, <) : J \text{ convex, } X \upharpoonright_J \cong M_{\beta, \gamma} \text{ for some } \beta \in I_\alpha, \gamma < \lambda\}$$

and similarly

$$Occ_\alpha^-(X) := \{J \subseteq (\lambda, <) : J \text{ convex, } X \upharpoonright_J \cong M_{\beta, \gamma}^{-1} \text{ for some } \beta \in I_\alpha, \gamma < \lambda\}.$$

As in the proof of Theorem 2.4 we can see that the sets  $Occ_\alpha^+(X)/\sim_\lambda$  and  $Occ_\alpha^-(X)/\sim_\lambda$  are finite for any reduced word  $X$  and  $\alpha \in 2^{2^\lambda}$ . Moreover, the maps  $\varphi_\alpha : \mathbb{X}_{\beta \in \lambda} G_\beta \longrightarrow \mathbb{Z}$  defined by

$$X \mapsto |Occ_\alpha^+(V)/\sim_\lambda| - |Occ_\alpha^-(V)/\sim_\lambda|$$

where  $V$  is the reduced word corresponding to  $X$ , are well-defined homomorphisms and  $\varphi_\alpha \neq \varphi_\beta$  for  $\alpha, \beta \in 2^{2^\lambda}$  since the families  $F_\delta$  are almost disjoint and satisfy condition (\*). Hence the size of all homomorphisms from the complete free product of the  $G_\alpha$ 's to the integers is  $2^{2^\lambda}$  as claimed. It remains to show the existence of the words  $M_\alpha$  satisfying (\*).

We start with any partition of  $\lambda$  into two sets, i.e. with a function  $g : \lambda \rightarrow \{0, 1\}$ . Note that there are  $2^\lambda$  of those functions. Moreover, we choose elements  $e_\alpha \neq h_\alpha \in G_\alpha$  ( $\alpha \in \lambda$ ) and define the word  $M'_g \in \ast_{\alpha \in \lambda} G_\alpha$  by

$$M'_g(\beta) = h_{\beta+2g(\beta)}$$

Then  $M'_g$  is a reduced word and we let  $M_g$  be the composition of  $M'_g$  with itself  $\omega_1$  times. Then  $M_g$  is still reduced and for different  $g, g' : \lambda \rightarrow \{0, 1\}$  condition (\*) is satisfied for  $M_g$  and  $M_{g'}$ . Thus the family

$$F = \{M_g : g : \lambda \rightarrow \{0, 1\}\}$$

is a family of reduced words of size  $2^\lambda$  satisfying (\*) as desired.  $\square$

Moreover, the authors would like to mention that modifying the proof of Theorem 2.4 Conner and Eda proved a more general result in [CE]

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